Q5, January 2018

V = P_3(C) = {polynomials of degree at most 3 with coefficients in C}
Find the Jordan Canonical Form of the linear operator T defined by T(f) = f + f''.

V = span_C{1, x, x^2, x^3}
A is the matrix of T with respect to the basis {1, x, x^2, x^3}.

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Strategy:

Find the Smith Normal Form of xI - A. Call it SNF(xI - A). The Smith Normal Form is a diagonal matrix whose diagonal entries satisfy d_1 | d_2 | … | d_n. The non-constant diagonal entries are called the invariant factors of A. The largest invariant factor d_n is the minimal polynomial of A. The product of all invariant factors of A is the characteristic polynomial of A.

How does one find the Smith Normal Form of xI - A?

We are allowed to use elementary row and column operations.

- We may take any pair of rows (or columns) and interchange them, e.g., $R_i \leftrightarrow R_j$.
- We may take any multiple of a row (or column) and add (or subtract) it to another row (or column), replacing that row (or column) by the new quantity, e.g., $cR_i + R_j \mapsto R_j$.

$$xI - A = \begin{pmatrix} x-1 & 0 & -2 & 0 \\ 0 & x-1 & 0 & -6 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{pmatrix}$$

$*\det(xI - A) = (x-1)^4$

$$\sim \begin{pmatrix} -2 & 0 & x-1 & 0 \\ 0 & -6 & 0 & x-1 \\ x-1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \end{pmatrix}$$

$C_1 \leftrightarrow C_3$

$C_2 \leftrightarrow C_4$

$$\sim \begin{pmatrix} 1 & 0 & x-1 & 0 \\ 0 & 1 & 0 & x-1 \\ -\frac{1}{2}(x-1) & 0 & 0 & 0 \\ 0 & -\frac{1}{6}(x-1) & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{2}C_1 \mapsto C_1$$

$$-\frac{1}{6}C_2 \mapsto C_2$$

$$\sim \begin{pmatrix} 1 & 0 & x-1 & 0 \\ 0 & 1 & 0 & x-1 \\ \boxed{0} & 0 & \boxed{\frac{1}{2}(x-1)^2} & 0 \\ 0 & \boxed{0} & 0 & \boxed{\frac{1}{6}(x-1)^2} \end{pmatrix}$$

$$\boxed{\frac{1}{2}(x-1)R_1 + R_3 \mapsto R_3}$$

$$\boxed{\frac{1}{6}(x-1)R_2 + R_4 \mapsto R_4}$$

$$\sim \begin{pmatrix} 1 & 0 & \boxed{0} & 0 \\ 0 & 1 & 0 & \boxed{0} \\ 0 & 0 & \boxed{(x-1)^2} & 0 \\ 0 & 0 & 0 & \boxed{(x-1)^2} \end{pmatrix}$$

$$\boxed{-(x-1)C_1 + C_3 \mapsto C_3}$$

$$\boxed{2C_3 \mapsto C_3}$$

$$\boxed{-(x-1)C_2 + C_4 \mapsto C_4}$$

$$\boxed{6C_4 \mapsto C_4}$$

invariant factors: (x - 1)^2 and (x - 1)^2

minimal polynomial: (x - 1)^2 (the largest invariant factor)

characteristic polynomial: (x - 1)^2 (x - 1)^2 = (x - 1)^4

$$SNF(xI - A) = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & (x-1)^2 & \\ 0 & & & (x-1)^2 \end{pmatrix}$$

For the Rational Canonical Form, we need the invariant factors.

$$RCF(A) = \boxed{C_{(x-1)^2}} \oplus \boxed{C_{(x-1)^2}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

RCF(A) = direct sum of companion matrices of invariant factors of A

$$C_{f(x)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & 1 & \cdots & 0 & -a_3 \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

I_{n-1}

$$f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$$

For the Jordan Canonical Form, we need the elementary divisors.

In general, the elementary divisors are found from the non-constant invariant factors. Explicitly, for $d_i(x) = p_1(x)^{e_1} \ldots p_k(x)^{e_k}$, the elementary divisors are the $p_j(x)^{e_j}$.

$d_3(x) = (x-1)^2$ —— elementary divisor

$d_4(x) = (x-1)^2$ —— elementary divisor

$d(x) = (x - 1)(x + 2)(x + 3)^2$

elementary divisors: $x - 1$, $x + 2$, $(x + 3)^2$

$d(x) = (x^2 - 4)(x^2 + 1)$

elementary divisors: $x - 2$, $x + 2$, $x^2 + 1$

If the field over which the vector space is defined is algebraically closed, then $x^2 + 1 = (x - i)(x + i)$, hence the elementary divisors are $x - 2$, $x + 2$, $x - i$, $x + i$.

$$JCF(A) = J_{(x-1)^2} \oplus J_{(x-1)^2} = \begin{pmatrix} 1 & 1 & & 0 \\ 0 & 1 & & \\ & & 1 & 1 \\ 0 & & 0 & 1 \end{pmatrix}$$

JCF(A) is the direct sum of the Jordan blocks of the elementary divisors.

$$J_{(x-a)^k} = \begin{pmatrix} a & 1 & & 0 \\ & a & \ddots & \\ & & \ddots & 1 \\ 0 & & & a \end{pmatrix} = aI + \text{matrix of 1s on the superdiagonal, 0s elsewhere}$$